Dept. of Math. University of Oslo
Pure Mathematics No. 25
ISSN 0806–2439 September 2004

POSITIVE SOLUTIONS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH SINGULAR NONLINEAR TERM*

Giuseppe Maria Coclite
C.M.A. (Centre of Mathematics for Applications), University of Oslo,
P.O. Box 1053 Blindern, 0316 Oslo, NORWAY
e-mail address: giusepc@math.uio.no

and

Mario Michele Coclite
Dipartimento di Matematica, Università di Bari,
via Orabona 4, 70125 Bari, ITALY
e-mail address: coclite@dm.uniba.it

Abstract. The existence of a positive solution in a weighted Sobolev space for an homogeneous semilinear elliptic integro-differential Dirichlet problem is proved. The integral operator of the equation depends on a nonlinear function with a singularity at the origin.

1. Introduction.

In this paper we establish an existence result for the following integro-differential problem

$$\begin{cases} -\Delta u(y) = \displaystyle\int_\Omega K(y,z)g\big(z,u(z)\big)dz, & \text{for } y \in \Omega, \\ u(y) = 0, & \text{for } y \in \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N$, $N \geq 3$, open bounded sufficiently smooth and $g(z,s)$, $z \in \Omega$, $s > 0$, bounded in a neighborhood of $+\infty$ and possibly nonsmooth as $s \to 0^+$; in particular we do not

* Work supported by M.U.R.S.T. Italy (fondi 40%, 60%) and by G.N.A.M.P.A. of I.N.dA.M.
2000 *Mathematics Subject Classification*: 45E99; 45G10; 45L99.
Key words and phrases: integro-differential equations, singular nonlinearity, existence of positive solutions.

exclude that

$$\underline{\lim}_{s\to 0^+} g(y,s)=0; \qquad \overline{\lim}_{s\to 0^+} g(y,s)=+\infty.$$

We do not assume anything about the existence of super or sub solutions. More precisely, denoting

$$\delta(x):=\operatorname{dist}(x,\partial\Omega), \quad x\in\mathbb{R}^N,$$

we shall assume

($\mathcal{A}_1$) $g:\Omega\times\mathbb{R}_+^*\to\mathbb{R}$ *is a Carathèodory function (namely* $g(\cdot,s)$ *is measurable in* Ω *for all* $s>0$; $g(z,\cdot)$ *is continuous in* $\mathbb{R}_+^*$ *for almost all* $z\in\Omega$ *) such that*

$$0\le g(z,s)\le \frac{\varphi_0(z)}{s^p}, \quad z\in\Omega, \quad 0<s\le\frac{1}{2}, \quad p\ge\frac{N}{N-1},$$

where $\varphi_0\in L^p(\Omega)$ *is a nonnegative map such that*

$$\frac{\varphi_0}{\delta^{p-1}}\in L^p(\Omega).$$

Moreover, $g^*(\cdot,s)\in L^p(\Omega)$, $s>0$, *where*

$$g^*(z,s):=\sup_{s\le t} g(z,t), \quad (z,s)\in\Omega\times\mathbb{R}_+^*,$$

(that is a Carathéodory function).

($\mathcal{A}_2$) $K\in L^q(\Omega\times\Omega)$, $q>N$, *is a nonnegative nucleus such that*

$$\frac{\delta(z)}{c_0}\le\int_\Omega K(y,z)\delta(y)dy\le c_0\delta(z), \quad z\in\Omega,$$

for some positive constant c_0.

($\mathcal{A}_3$) *There exist* $\mu_0>0$ *and* $\Omega_0\subset\Omega$, $|\Omega_0|>0$, *such that*

$$\underline{\lim}_{s\to 0}\frac{g(z,s)}{s}\ge\mu_0, \quad uniformly\ with\ respect\ to\ z\in\Omega_0.$$

Due to the assumption ($\mathcal{A}_1$), assuming the existence of a subsolution, the existence of solutions to (1.1) is trivial.

We prove that if μ_0 is bigger than the smallest characteristic value of the operator

$$\varphi\mapsto\int_{\Omega_0} H(x,\cdot)\varphi(x)dx, \quad \varphi\in L^1(\Omega_0),$$

there exists a weak solution $u_0 \in L^1(\Omega)$ to (1.1), that is positive a. e. in Ω, $\delta|\nabla u| \in L^1(\Omega)$ and with trivial trace on $\partial\Omega$.

Our arguments use the properties of the Green's function $G(x, y)$ associated to $-\Delta$ in Ω with homogeneous conditions on $\partial\Omega$ and the ones of the nucleus

$$H(x, z) := \int_\Omega G(x, y)K(y, z)dy.$$

In the first part of the paper we look for an existence result for the integral equation of Hammerstein type

$$u(x) = \int_\Omega H(x, z)g(z, u(z))dz. \tag{1.2}$$

The argument is based on the results of the papers [5; 6; 7], where references and applications for this type of equations can be found.

Integro-differential problems like (1.1) are present in the literature (see for example [12; 13; 15] and the references therein).

The paper is organized as follows: §2. Notations and results. §3. Properties of the nuclei G, K, H. §4. Proofs of Theorems 1 and 2. §5. On the integral equation (1.2). §6. Proof of Theorem 3.

2. Notations and results.

Let us list the notations mostly used in this paper.

$\mathbb{R}_+ := [0, +\infty[$; $\mathbb{R}_+^* :=]0, +\infty[$; $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Let $E \subset \mathbb{R}^N$ be a measurable set, $|E|$ is the measure of E, $|\cdot|_{q,E}$ is the $L^q(E)-$norm and $L^q_+(E)$ is the cone of the $\varphi \in L^q(E)$, $\varphi \geq 0$ a. e. in E. $L^1(\delta, E)$ is the set of the φ such that $\delta\varphi \in L^1(E)$, $L^1_+(\delta, E)$ is the cone of the $\varphi \geq 0$ a. e. in E such that $\delta\varphi \in L^1(E)$ and $W^{1,1}(\delta, E)$ is the space of the $\varphi \in L^1(E)$ with the modulus of the gradient (in the sense of distributions) belonging to $L^1_+(\delta, E)$. $W^{1,1}_0(\delta, E)$ is the subspace of the $\varphi \in W^{1,1}(\delta, E)$ with trivial trace on ∂E.

Let u, v be two maps, $u \leq v$ is the set of the points $x \in \Omega$ such that $u(x) \leq v(x)$. Analogously, we define $u < v$, $u \geq v$, $u > v$.

Finally, $D = \text{diam}(\Omega)$, $B_R(x)(\subset \mathbb{R}^N)$ is the ball centered in x with radius R and σ_N is the $(N-1)-$dimensional measure of $\partial B_1(0)$.

Let $E \subset \Omega$ be a measurable set, define

$$\lambda(E) := \inf\left\{\lambda(E, \varphi) \,\middle|\, \varphi \in L^1_+(\Omega),\, \varphi \neq 0\right\},$$

where

$$\lambda(E,\varphi)=\sup_{z\in E^*(\varphi)}\frac{\varphi(z)\chi_E(z)}{\int\limits_E H(x,z)\varphi(x)dx};\quad E^*(\varphi)=\Big\{z\in E\Big|\int\limits_E H(x,z)\varphi(x)dx\neq 0\Big\}.$$

The main results of this paper are the following.

Theorem 1 *Let* $E\subset\Omega$ *be a measurable set,* $|E|>0$. $\lambda(E)$ *is the smallest positive characteristic value of the operator*

$$\varphi\mapsto\int\limits_E H(x,\cdot)\varphi(x)dx,\quad \varphi\in L^1(\Omega).$$

Useful for the following Theorem 3 is the left continuity of $\lambda(E)$.

Theorem 2 $\lambda(\cdot)$ *is left continuous, more precisely, for each measurable set* $E\subset\Omega$, $|E|>0$, *and* $\alpha>0$ *there exists* $\sigma>0$ *such that for every measurable set* $F\subset E$ *there results*

$$|E\setminus F|<\sigma\ \Rightarrow\ \lambda(E)\le\lambda(F)\le\lambda(E)+\alpha.$$

Other properties of $\lambda(E)$ are proved in Section 4. Finally, as said in the Introduction, the following result holds.

Theorem 3 *Assume* $(\mathcal{A}_i)$, $i=1,\ 2,\ 3$ *and*

$$\mu_0>\lambda(\Omega_0).$$

There exists $u_0\in W_0^{1,1}(\delta,\Omega)$, $u_0>0$ *a. e. in* Ω, *weak solution to (1.1).*

3. Properties of the nuclei G, K, H.

In this section we prove some properties of the nuclei G, K, H and of the associated integral operators that are crucial in the proofs of Theorems 1, 2, 3.

The exponent q present in the following statements is the one of $(\mathcal{A}_2)$ and q' is the conjugate one.

Lemma 3.1 *There exists* $c_1>0$ *such that, for each* $x,y\in\Omega$, $x\neq y$, *there results*

$$\frac{1}{c_1|x-y|^{N-2}}\le G(x,y)\le\frac{c_1}{|x-y|^{N-2}},\quad |x-y|\to 0. \tag{3.1}$$

$$(3.2) \qquad |\nabla_x G(x,y)| \le \frac{c_1}{|x-y|^{N-1}}.$$

$$(3.3) \qquad |\nabla_x G(x,y)| \le \frac{c_1 \delta(y)}{|x-y|^{N}}.$$

$$(3.4) \qquad |\delta(x)\nabla_x G(x,y)| \le \frac{c_1 \delta(y)}{|x-y|^{N-1}}.$$

$$(3.5) \qquad \frac{\delta(x)\delta(y)}{c_1} \le G(x,y); \qquad \int_\Omega G(x,y)dx \le c_1\delta(y).$$

$$(3.6) \qquad \Big(\int_\Omega G(x,y)^r dy\Big)^{\frac{1}{r}} \le c_1 \int_\Omega G(x,y)dy, \quad 1 \le r < \frac{N}{N-1}.$$

$$(3.7) \qquad \Big(\int_\Omega G(x,y)^{\frac{N}{N-1}} dy\Big)^{\frac{N-1}{N}} \le c_1\delta(x)|\log\delta(x)|.$$

Proof. (3.1) is wellknown (see for example [1, Chapter 4]). (3.2) and (3.3) are proved in [10; 14]. (3.4) is consequence of these ones. (3.5) is proved in [3, Lemma 3.2; 4, Theorem 9; 16, Theorem 1]. Finally, (3.6) and (3.7) are shown in [2, Theorem 1 and (1.9)].■

Lemma 3.2 *There results*

$$\forall s > 0: \quad K(y,\cdot)g^*(\cdot,s) \in L^1(\Omega), \ \text{ a.e. } y \in \Omega; \qquad \int_\Omega K(\cdot,z)g^*(z,s)dz \in L^q(\Omega).$$

Proof. The claim follows from $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$.■

Lemma 3.3 *The following statements are equivalent*

$$i) \qquad \exists c_0 > 0: \quad \frac{\delta(z)}{c_0} \le \int_\Omega K(y,z)\delta(y)dy \le c_0\delta(z), \quad z \in \Omega.$$

$$ii) \qquad \exists c_2 > 0: \quad \frac{\delta(x)\delta(z)}{c_2} \le H(x,z); \quad \int_\Omega H(x,z)dx \le c_2\delta(z), \quad x,z \in \Omega.$$

Proof. *i)* $\Rightarrow$ *ii)* Trivial consequence of the definition of $H(x,z)$ and (3.5).

Proof. *ii)* $\Rightarrow$ *i)* Let $\varphi_1(x)$ be a positive eigenfunction and λ_1 the first eigenvalue of the Dirichlet problem for $-\Delta$ on Ω. We have that

$$\lambda_1 \int_\Omega H(x,z)\varphi_1(x)dx = \lambda_1 \int_\Omega K(y,z)dy \int_\Omega G(x,y)\varphi_1(x)dx = \int_\Omega K(y,z)\varphi_1(y)dy.$$

By Theorem 9 in [4], there exists $c_3 > 0$ such that

$$\frac{\delta(x)}{c_3} \le \varphi_1(x) \le c_3\delta(x).$$

Therefore, using $ii)$,

$$\frac{\lambda_1\delta(z)}{c_2}\int\limits_\Omega \delta(x)\varphi_1(x)dx \le c_3\int\limits_\Omega K(y,z)\delta(y)dy$$

and

$$\frac{1}{c_3}\int\limits_\Omega K(y,z)\delta(y)dy \le \lambda_1|\varphi_1|_{\infty,\Omega}\int\limits_\Omega H(x,z)dx \le \lambda_1 c_2|\varphi_1|_{\infty,\Omega}\delta(z).$$

Then $i)$ is proved.■

Theorem 3.4 *The following statements hold*

(3.8) $\quad H : \varphi \mapsto \int\limits_\Omega H(\cdot,z)\varphi(z)dz$ *is bounded from* $L^1(\delta,\Omega)$ *in* $L^1(\Omega)$.

(3.9) $\quad \tilde{H} : \varphi \mapsto \int\limits_\Omega H(x,\cdot)\varphi(x)dx$ *is bounded from* $L^1(\delta,\Omega)$ *in* $L^q(\Omega)$.

(3.10) $\quad$ *For each* $s > 0$: $(x,z) \mapsto H(x,z)g^*(z,s)$ *belongs to* $L^1(\Omega\times\Omega)$.

Proof. (3.8) Let $\varphi \in L^1(\delta,\Omega)$. From $ii)$ of Lemma 3.3,

$$|H(\varphi)|_{1,\Omega} \le \int\limits_\Omega |\varphi(z)|dz\int\limits_\Omega H(x,z)dx \le c_2\int\limits_\Omega |\varphi(z)|\delta(z)dz = c_2|\delta\varphi|_{1,\Omega}.$$

Proof. (3.9) Let $\varphi \in L^1(\delta,\Omega)$. Since $q' < \frac{N}{N-1}$, by (3.6) and (3.5),

$$|\tilde{H}(\varphi))|^q_{q,\Omega} = \int\limits_\Omega |\tilde{H}(\varphi)(z)|^q dz = \int\limits_\Omega dz|\int\limits_\Omega \varphi(x)dx\int\limits_\Omega G(x,y)K(y,z)dy|^q \le \tag{3.11}$$

$$\le \int\limits_\Omega dz\left\{\int\limits_\Omega |\varphi(x)|dx\big(\int\limits_\Omega G(x,y)^{q'}dy\big)^{\frac{1}{q'}}\big(\int\limits_\Omega K(y,z)^q dy\big)^{\frac{1}{q}}\right\}^q \le$$

$$\le c_1^{2q}\int\limits_\Omega dz\left\{\int\limits_\Omega |\varphi(x)\delta(x)|dx\big(\int\limits_\Omega |K(y,z)|^q dy\big)^{\frac{1}{q}}\right\}^q = c_1^{2q}|K|^q_{q,\Omega\times\Omega}|\varphi\delta|^q_{1,\Omega}.$$

Proof. (3.10) From $(\mathcal{A}_1)$ we have $g^*(\cdot,s) \in L^1(\Omega)$, hence, using (3.8), (3.10) is consequence of the Tonelli Theorem.■

Theorem 3.5 *H is compact from $L^1(\delta,\Omega)$ in $L^1(\Omega)$.*

Proof. We claim that H is the limit of a sequence of linear compact operators from $L^1(\delta,\Omega)$ in $L^1(\Omega)$.

Let

$$\widetilde{D} := \big\{(x,x) \,|\, x \in \mathbb{R}^N\big\}$$

be the diagonal set of $\mathbb{R}^N \times \mathbb{R}^N$. Remind that the Green's function $G(x,y)$ is strictly positive in $\Omega\times\Omega$, continuous in $(\bar{\Omega}\times\bar{\Omega})\setminus\widetilde{D}$, vanishes on $\partial(\Omega\times\Omega)\setminus\widetilde{D}$ and, since $N>1$,

$$\lim_{|x-y|\to 0} G(x,y) = +\infty$$

(see [1, Chapter 4]). Let $n\in\mathbb{N}$, define

$$G_n(x,y) := \begin{cases} \dfrac{nG(x,y)}{n+G(x,y)}, & \text{for } x\neq y,\\ n, & \text{for } x=y. \end{cases}$$

Clearly $G_n \le G$, $G_n \in C(\bar{\Omega}\times\bar{\Omega})$, G_n is strictly positive in $\Omega\times\Omega$ and vanishes on $\partial(\Omega\times\Omega)$.

Consider the linear operator

$$H_n(\varphi) := \chi_{\Omega_n}(\cdot)\int\limits_{\Omega} H_n(\cdot,z)\varphi(z)dz, \quad \varphi\in L^1(\delta,\Omega),$$

where

$$\Omega_n = \{x\in\Omega \,|\, \delta(x) \ge \frac{1}{n}\}, \qquad H_n(x,z) := \int_{\Omega} G_n(x,y)K(y,z)dy.$$

Since $G_n \le G_{n+1} \le G$, H_n is continuous from $L^1(\delta,\Omega)$ in $L^1(\Omega)$ and

$$\|H_n\| \le \|H_{n+1}\| \le \|H\|. \tag{3.12}$$

The claim is consequence of the following lemmas.

Lemma 3.6 *H_n is compact from $L^1(\delta,\Omega)$ in $L^1(\Omega)$.*

Proof. Let $\mathcal{F}\subset L^1(\delta,\Omega)$ be bounded, by (3.8) and (3.12), $H_n(\mathcal{F})$ is bounded in $L^1(\Omega)$. We prove the equicontinuity of $H_n(\varphi)$, $\varphi\in\mathcal{F}$, in $L^1(\Omega)$,

$$\Delta(h,\varphi) = |H_n(\varphi)(\cdot+h) - H_n(\varphi)|_{1,\Omega} \le$$

$$\leq \int\limits_{\Omega} \chi_{\Omega_n}(x+h)dx \int\limits_{\Omega} |\big(H_n(x+h,z)-H_n(x,z)\big)\varphi(z)|dz+$$

$$+\int\limits_{\Omega} |\chi_{\Omega_n}(x+h)-\chi_{\Omega_n}(x)|dx \int\limits_{\Omega} H_n(x,z)|\varphi(z)|dz.$$

Assume that $|h| \leq \dfrac{1}{2n}$, observe

$$\big((x+h)\in\Omega_n \text{ and } x\in\Omega\big) \Rightarrow \frac{1}{n} \leq \delta(x+h) \leq \delta(x)+|h| \Rightarrow \frac{1}{2n} \leq \delta(x),$$

and

$$\big(|\chi_{\Omega_n}(x+h)-\chi_{\Omega_n}(x)| = 1 \text{ and } x\in\Omega\big) \Rightarrow$$
$$\Rightarrow \big((x+h)\in\Omega_n \text{ and } x\notin\Omega_n\big) \vee \big(x\in\Omega_n \text{ and } (x+h)\notin\Omega_n\big) \Rightarrow$$
$$\Rightarrow \big(\delta(x) < \frac{1}{n} \leq \delta(x+h)\big) \vee \big(\delta(x+h) < \frac{1}{n} \leq \delta(x)\big) \Rightarrow$$
$$\Rightarrow \big(\frac{1}{n}-|h| \leq \delta(x) < \frac{1}{n}\big) \vee \big(\frac{1}{n} \leq \delta(x) < \frac{1}{n}+|h|\big) \Rightarrow \big(\frac{1}{n}-|h| \leq \delta(x) < \frac{1}{n}+|h|\big).$$

Denoting

$$E_h = \big\{x\in\Omega \,\big|\, \frac{1}{n}-|h| \leq \delta(x) < \frac{1}{n}+|h|\big\},$$

we have

$$\lim_{h\to 0} |E_h| = 0,$$

and

$$\Delta(h,\varphi) \leq \int\limits_{\Omega_{2n}} dx \int\limits_{\Omega} |H_n(x+h,z)-H_n(x,z)|\cdot|\varphi(z)|dz+$$

$$+\int\limits_{E_h} dx \int\limits_{\Omega} H_n(x,z)|\varphi(z)|dz = \Delta_1(h,\varphi)+\Delta_2(h,\varphi).$$

We estimate $\Delta_1(h,\varphi)$, $\Delta_2(h,\varphi)$. Since

$$\Delta_1(h,\varphi) \leq \int\limits_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz \int\limits_{\Omega_{2n}} |G_n(x+h,y)-G_n(x,y)|dx$$

and

$$x\in\Omega_{2n},\ |h| < \frac{1}{2n} \Rightarrow x+th\in\Omega,\ 0\leq t\leq 1,$$

there results

$$|h| < \frac{1}{2n} \Rightarrow \gamma(h,y) := \int\limits_{\Omega_{2n}} |G_n(x+h,y)-G_n(x,y)|dx =$$

$$= \int_{\Omega_{2n}} dx \Big| \int_0^1 \frac{d}{dt} G_n(x+th,y)dt \Big| = \int_{\Omega_{2n}} dx \Big| \int_0^1 \frac{n^2 \nabla_x G(x+th,y)\cdot h}{(n+G(x+th,y))^2} dt \Big|.$$

From (3.1) and (3.3),

$$|h| < \frac{1}{2n} \Rightarrow \gamma(h,y) \le n^2 |h| \int_{\Omega_{2n}} dx \int_0^1 \frac{\frac{c_1 \delta(y)}{|x+th-y|^N}}{\left(n + \frac{1}{c_1 |x+th-y|^{N-2}}\right)^2} dt \le$$

$$\le n^2 c_1^3 |h| \delta(y) \int_0^1 dt \int_\Omega \frac{|x+th-y|^{N-4}}{(nc_1|x+th-y|^{N-2}+1)^2} dx \le$$

$$\le n^2 c_1^3 |h| \delta(y) \int_0^1 dt \int_{B_D(y-th)} \frac{|x+th-y|^{N-4}}{(nc_1|x+th-y|^{N-2}+1)^2} dx =$$

$$= n^2 c_1^3 |h| \delta(y) \sigma_N \int_0^D \frac{\rho^{N-4}\cdot \rho^{N-1}}{(nc_1 \rho^{N-2}+1)^2} d\rho \le n^2 c_1^3 \sigma_N |h| \delta(y) \frac{D^{2N-4}}{2N-4},$$

then, there exists $c > 0$, independent on h and y, such that

$$|h| < \frac{1}{2n} \Rightarrow \int_{\Omega_{2n}} |G_n(x+h,y) - G_n(x,y)| dx \le c|h|\delta(y).$$

Due to $(\mathcal{A}_2)$,

$$\Delta_1(h,\varphi) \le c|h| \int_{\Omega\times\Omega} \delta(y) K(y,z) |\varphi(z)| dy dz \le cc_0 |h| \int_\Omega \delta(z)|\varphi(z)| dz.$$

Let $|h| < 1/(2n)$, using the Hölder inequality, $(\mathcal{A}_2)$, (3.6) and (3.5),

$$\Delta_2(h,\varphi) \le \int_{\Omega\times\Omega} K(y,z)|\varphi(z)| dy dz \int_{E_h} G(x,y) dx \le$$

$$\le \int_{\Omega\times\Omega} K(y,z)|\varphi(z)| dy dz \Big(\int_\Omega G(x,y)^{q'} dx \Big)^{\frac{1}{q'}} |E_h|^{\frac{1}{q}} \le$$

$$\le c_1^2 |E_h|^{\frac{1}{q}} \int_{\Omega\times\Omega} \delta(y) K(y,z) |\varphi(z)| dy dz \le c_0 c_1^2 |E_h|^{\frac{1}{q}} \int_\Omega |\varphi(z)| \delta(z) dz.$$

Thanks to the estimates on $\Delta_1(h,\varphi)$, $\Delta_2(h,\varphi)$,

$$|h| \le \frac{1}{2n} \Rightarrow |H_n(\varphi)(\cdot + h) - H_n(\varphi)|_{1,\Omega} \le \big(cc_0|h| + c_0c_1^2|E_h|^{\frac{1}{q}}\big) \int\limits_{\Omega} |\varphi(z)|\delta(z)dz.$$

Then, $H_n(\varphi)$, $\varphi \in \mathcal{F}$, is equicontinuous in $L^1(\Omega)$. Due to the Frechet-Kolmogorov Theorem $H_n(\mathcal{F})$ is relatively compact in $L^1(\Omega)$, this proves the compactness of H_n.■

Lemma 3.7 *$H_n \to H$ in the operator norm.*

Proof. Let $\varphi \in L^1(\delta,\Omega)$, $|\delta\varphi|_{1,\Omega} = 1$. There results

$$\Lambda_n(\varphi) = |H(\varphi) - H_n(\varphi)|_{1,\Omega} = \int\limits_{\Omega} dx| \int\limits_{\Omega} \big(H(x,z) - \chi_{\Omega_n}(x)H_n(x,z)\big)\varphi(z)dz| \le$$

$$\le \int\limits_{\Omega\setminus\Omega_n} dx \int\limits_{\Omega} H(x,z)|\varphi(z)|dz + \int\limits_{\Omega_n} dx \int\limits_{\Omega} |H(x,z) - H_n(x,z)||\varphi(z)|dz =$$

$$= \int\limits_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\big(\int\limits_{\Omega\setminus\Omega_n} G(x,y)dx + \int\limits_{\Omega_n} |G(x,y) - \frac{nG(x,y)}{n+G(x,y)}|dx\big) = \Lambda_n'(\varphi) + \Lambda_n''(\varphi).$$

Using the Hölder inequality, (3.5), (3.6) and $(\mathcal{A}_2)$, we get

$$\Lambda_n'(\varphi) \le \int\limits_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\big(\int\limits_{\Omega\setminus\Omega_n} G(x,y)^{q'}dx\big)^{\frac{1}{q'}} |\Omega \setminus \Omega_n|^{\frac{1}{q}} \le$$

$$\le c_1^2|\Omega\setminus\Omega_n|^{\frac{1}{q}} \int\limits_{\Omega} |\varphi(z)|dz \int\limits_{\Omega} \delta(y)K(y,z)dy \le c_0c_1^2|\Omega\setminus\Omega_n|^{\frac{1}{q}} \int\limits_{\Omega} |\varphi(z)|\delta(z)dz \le c_0c_1^2|\Omega\setminus\Omega_n|^{\frac{1}{q}}.$$

Again using the Hölder inequality, (3.7) and (3.5),

$$\Lambda_n''(\varphi) \le \int\limits_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz \left(\int\limits_{\Omega_n} G(x,y)^{\frac{N}{N-1}}dx \right)^{\frac{N-1}{N}} \left(\int\limits_{\Omega_n} \big(\frac{G(x,y)}{n+G(x,y)}\big)^N dx \right)^{\frac{1}{N}} \le$$

$$\le c_1 \int\limits_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)|\ln\delta(y)|dydz \left(\int\limits_{\Omega} \frac{G(x,y)}{n+G(x,y)}dx \right)^{\frac{1}{N}} \le$$

$$\le \frac{c_1}{\sqrt[N]{n}} \int\limits_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)|\ln\delta(y)|dydz \left(\int\limits_{\Omega} G(x,y)dx \right)^{\frac{1}{N}} \le$$

$$\leq \frac{c_1^{\frac{1+N}{N}}}{\sqrt[N]{n}} \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)^{\frac{1+N}{N}}|\ln\delta(y)|dydz.$$

Hence there exists $c>0$, independent on n, φ, such that, by $(\mathcal{A}_2)$,

$$\Lambda_n''(\varphi) \leq \frac{c}{\sqrt[N]{n}} \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)dydz \leq \frac{c_0 c}{\sqrt[N]{n}} \int_{\Omega} |\varphi(z)|\delta(z)dz \leq \frac{c_0 c}{\sqrt[N]{n}}.$$

Finally, from the estimates on $\Lambda_n'(\varphi)$, $\Lambda_n''(\varphi)$, there exists $c>0$, independent on n, φ, such that

$$|\delta\varphi|_{1,\Omega}=1 \Rightarrow |H(\varphi)-H_n(\varphi)|_{1,\Omega} \leq c\big(|\Omega\setminus\Omega_n|^{\frac{1}{q}}+\frac{1}{\sqrt[N]{n}}\big).$$

This proves the claim.■

Theorem 3.8 *$\tilde{H}$ is compact from $L^1(\delta,\Omega)$ in $L^q(\Omega)$.*

Proof. Let $\mathcal{F}\subset L^1(\delta,\Omega)$ be bounded, by (3.9), $\tilde{H}(\mathcal{F})$ is bounded in $L^q(\Omega)$. We prove the equicontinuity of $\tilde{H}(\varphi)$, $\varphi\in\mathcal{F}$, in $L^q(\Omega)$. Arguing as in (3.11)

$$|\tilde{H}(\varphi)(\cdot+h)-\tilde{H}(\varphi)|_{q,\Omega}^q \leq c_1^{2q}|\varphi\delta|_{1,\Omega}^q \cdot \int_{\Omega\times\Omega} |K(y,z+h)-K(y,z)|^q dydz.$$

Therefore, the equicontinuity of $\tilde{H}(\varphi)$, $\varphi\in\mathcal{F}$, is consequence of the boundedness of $\mathcal{F}$ in $L^1(\delta,\Omega)$ and of the $L^q(\Omega\times\Omega)-$mean continuity of K. Finally, the compactness of $\tilde{H}$ is consequence of the Frechet-Kolmogorov Theorem.■

Corollary 3.9 *Let $E\subset\Omega$ be a measurable set, $|E|>0$. The operator*

$$\tilde{H}_E(\varphi) := \int_E H(x,\cdot)\varphi(x)dx, \quad \varphi\in L^1(\delta,E)$$

is compact from $L^1(\delta,E)$ in $L^q(E)$.

4. Proofs of Theorems 1 and 2.

Let $E\subset\Omega$ be measurable, $|E|>0$. The following lemmas are needed.

Lemma 4.1 *For every $\varphi\in L_+^1(E)$ there results*

(4.1) $$\int_E H(x,z)\varphi(x)dx \geq \frac{\delta(z)}{c_2}\int_E \delta(x)\varphi(x)dx.$$

(4.2) $\qquad E^*(\varphi) = E \setminus \partial\Omega, \ \ \varphi \neq 0.$

(4.3) $$\frac{1}{c_1^2 |E|^{\frac{1}{q'}} |K|_{q,\Omega\times\Omega} \cdot \sup\limits_E \delta} \leq \lambda(E) \leq \frac{c_2}{|\delta|_{2,E}^2}.$$

Proof. (4.1) is direct consequence of Lemma 3.3.ii).

Let $\varphi \in L^1_+(E)$, $\varphi \neq 0$, clearly

$$\int\limits_E \delta(x)\varphi(x)dx > 0.$$

Hence, (4.2) follows from (4.1).

We prove (4.3). Since $\delta > 0$, by the definition of $\lambda(E)$, (4.1) and (4.2),

$$\lambda(E) \leq \lambda(E,\delta) = \operatorname*{esssup}_{z\in E^*(\delta)} \frac{\delta(z)\chi_{E(z)}}{\int\limits_E H(x,z)\delta(x)dx} \leq \operatorname*{esssup}_{z\in E^*(\delta)} \frac{c_2\delta(z)}{\delta(z)\int\limits_E \delta(x)^2dx} = \frac{c_2}{|\delta|_{2,E}^2}.$$

Moreover, for each $\varphi \in L^1_+(E)$, $\varphi \neq 0$, using the definition of $\lambda(E,\varphi)$,

$$\int\limits_E \varphi(z)dz \leq \lambda(E,\varphi)\int\limits_E \varphi(x)dx \int\limits_E H(x,z)dz \leq$$

$$\leq \lambda(E,\varphi)\int\limits_E \varphi(x)dx \int\limits_E dz\Big(\int\limits_\Omega G(x,y)^{q'}dy\Big)^{\frac{1}{q'}}\Big(\int\limits_\Omega K(y,z)^q dy\Big)^{\frac{1}{q}}.$$

By (3.5) and (3.6),

$$\int\limits_E \varphi(z)dz \leq \lambda(E,\varphi)c_1^2\int\limits_E \varphi(x)\delta(x)dx \int\limits_E dz\Big(\int\limits_\Omega K(y,z)^q dy\Big)^{\frac{1}{q}} \leq$$

$$\leq \lambda(E,\varphi)c_1^2|E|^{\frac{1}{q'}}|K|_{q,\Omega\times\Omega}\cdot \sup_E \delta \cdot \int\limits_E \varphi(z)dz.$$

Since $0 < \int\limits_E \varphi(z)dz$,

$$\forall \varphi \in L^1_+(E),\ \varphi \neq 0: \qquad \frac{1}{c_1^2|E|^{\frac{1}{q'}}|K|_{q,\Omega\times\Omega}\cdot \sup\limits_E \delta} \leq \lambda(E,\varphi).$$

Again from the definition of $\lambda(E)$, we have the lower bound for $\lambda(E)$ stated in (4.3).■

Lemma 4.2 *There exists* $\Phi \in L^q_+(E)$, $\Phi > 0$ *a. e. in* E, *such that*

$$\lambda(E) = \operatorname*{esssup}_{z \in E \setminus \partial\Omega} \frac{\Phi(z)}{\int\limits_E H(x,z)\Phi(x)dx}.$$

Proof. Due to the definition of $\lambda(E)$, there exists $(\varphi_n)_{n\in\mathbb{N}^*}$, $|\delta\varphi_n|_{1,E} = 1$, such that

$$\operatorname*{esssup}_{z \in E \setminus \partial\Omega} \frac{\varphi_n(z)}{\tilde{H}_E(\varphi_n)(z)} < \lambda(E) + \frac{1}{n}. \tag{4.4}$$

Denoting $\tilde{H}_E(\varphi_n) = \Phi_n$, due to the compactness of $\tilde{H}_E$ from $L^1(\delta, E)$ in $L^q(E)$ (see Corollary 3.6) there exists $\Phi \in L^q(E)$, such that, passing to a subsequence,

$$\Phi_n \to \Phi, \quad \text{in } L^q(E).$$

From (4.1),

$$\Phi_n(z) \geq \frac{\delta(z)}{c_2}|\delta\varphi_n|_{1,E},$$

then $\Phi > 0$ a. e. in E. Moreover, since $H(\cdot,\cdot) \geq 0$, from (4.4),

$$\Phi_n = \tilde{H}_E(\varphi_n) \leq \big(\lambda(E) + \frac{1}{n}\big)\tilde{H}_E(\Phi_n) \quad \text{in } E.$$

By the continuity of $\tilde{H}_E$ (see Corollary 3.9),

$$\Phi \leq \lambda(E)\tilde{H}_E(\Phi), \quad \text{in } E,$$

hence, from the definition of $\lambda(E)$,

$$\operatorname*{esssup}_{z \in E \setminus \partial\Omega} \frac{\Phi(z)}{\tilde{H}_E(\Phi)(z)} = \lambda(E),$$

then, the proof is done. ▪

Proof of Theorem 1 We argue as in [11, Theorem 2.5]. Let $\Phi \in L^1_+(E)$, $|\Phi|_{1,E} = 1$, be a minimum point for $\varphi \mapsto \lambda(E,\varphi)$ (see Lemma 4.2). Consider the set

$$\mathcal{E} := \big\{\psi \in L^1(E) \,\big|\, |\psi|_{1,E} \leq 1,\ \psi \geq 0 \ a.e.\big\},$$

it is closed, bounded and convex. Moreover, the operator

$$A_n(\psi) := \frac{\tilde{H}_E(\psi + \frac{\Phi}{n})}{\big|\tilde{H}_E(\psi + \frac{\Phi}{n})\big|_{1,E}}, \quad n \in \mathbb{N}^*,$$

maps $\mathcal{E}$ in itself. The compactness of $\tilde{H}_E$ from $L^1(E)(\subset L^1(\delta,\Omega))$ in itself and the fact that

$$\forall \psi \in \mathcal{E}: \ \left|\tilde{H}_E(\psi + \frac{\Phi}{n})\right|_{1,E} \geq \frac{1}{n}\left|\tilde{H}_E(\Phi)\right|_{1,E} > 0$$

imply that $A_n(\mathcal{E})$ is compact. Due to the Shauder Fixed Point Theorem, there exists $\psi_n \in \mathcal{E}$ such that $A_n(\psi_n) = \psi_n$. Clearly, $|\psi_n|_{1,E} = 1$. Denoting

$$\mu_n = \frac{1}{\left|\tilde{H}_E(\psi_n + \frac{\Phi}{n})\right|_{1,E}},$$

we can rewrite the previous identity on ψ_n in the following way

$$\mu_n \tilde{H}_E(\psi_n + \frac{\Phi}{n}) = \psi_n. \tag{4.5}$$

Due to the positivity of $H(x,z)$ and Lemma 4.2,

$$\psi_n \geq \frac{\mu_n}{n}\tilde{H}_E(\Phi) \geq \frac{\mu_n}{\lambda(E)n}\Phi. \tag{4.6}$$

We claim that

$$\forall k \in \mathbb{N}: \quad \frac{\rho}{n}(1+\rho+\cdots+\rho^k)\Phi \leq \psi_n, \tag{4.7}$$

where

$$\rho = \frac{\mu_n}{\lambda(E)}.$$

The estimate for $k=0$ is the one stated in (4.6). For $k \geq 1$, observe that

$$\psi_n = \mu_n \tilde{H}_E(\psi_n + \frac{\Phi}{n}) \geq \mu_n \tilde{H}_E(\frac{\rho}{n}(1+\rho+\cdots+\rho^k)\Phi + \frac{\Phi}{n}) =$$

$$= \mu_n\big(\frac{\rho}{n}(1+\rho+\cdots+\rho^k) + \frac{1}{n}\big)\tilde{H}_E(\Phi) \geq \frac{\mu_n}{\lambda(E)n}(1+\rho+\cdots+\rho^{k+1})\Phi.$$

Arguing by induction we get (4.7). From (4.7), integrating on E,

$$\frac{\rho}{n}(1+\rho+\cdots+\rho^k) \leq 1, \quad k \in \mathbb{N}.$$

Then, $\rho < 1$, namely

$$\forall n \in \mathbb{N}^*: \quad \mu_n < \lambda(E). \tag{4.8}$$

By the compactness of $\tilde{H}_E$ (see Corollary 3.9) and the boundedness of $(\psi_n)_{n\in\mathbb{N}^*}$, there exist $(n_i)_{i\in\mathbb{N}}$, $n_i \to \infty$, $\Psi \in L^1(E)$, $\mu_0 \geq 0$, such that

$$\tilde{H}_E(\psi_{n_i} + \frac{\Phi}{n_i}) \longrightarrow \Psi \text{ in } L^1(E), \qquad \mu_0 = \lim_i \mu_{n_i}.$$

From (4.5),

$$\mu_0|\Psi|_{1,E} = 1,$$

hence $\mu_0 > 0$ and $\Psi \neq 0$. Again by (4.5), $(\psi_{n_i})_{i\in\mathbb{N}}$ converges to $\mu_0\Psi$, due to the continuity of $\tilde{H}_E$ (see Theorem 3.4),

$$\mu_0\tilde{H}_E(\Psi) = \Psi.$$

Using Lemma 3.3.*ii*), we get $\Psi \in L^1_+(E)$ and $\mu_0 = \lambda(E,\Psi)$. From the definition of $\lambda(E)$,

$$\lambda(E) \leq \mu_0,$$

and, by (4.8), we can conclude that: $\lambda(E) = \mu_0$. Finally, using again the definition of $\lambda(E)$, we have that μ_0 is the smallest characteristic value of $\tilde{H}_E$. ■

Lemma 4.3 *For each $\alpha > 0$ and $\varphi \in L^q_+(E)$ there exists $\sigma > 0$ such that for every measurable $F \subset E$ there results*

$$|E \setminus F| < \sigma \Rightarrow \int_{E\setminus F} H(x,z)\varphi(x)dx < \alpha \int_E H(x,z)\varphi(x)dx, \quad z \in \Omega.$$

Proof. We begin by observing that for each measurable $S \subset \Omega$

$$\int_S G(x,y)\varphi(x)dx \leq \Big(\int_\Omega G(x,y)^{q'}dx\Big)^{\frac{1}{q'}}\Big(\int_S \varphi(x)^q dx\Big)^{\frac{1}{q}}.$$

Since $q' < \dfrac{N}{N-1}$, due to the symmetry of G and (3.5), (3.6), we get

$$\int_S G(x,y)\varphi(x)dx \leq c_1^2|\varphi|_{q,S}\delta(y), \quad y \in \Omega. \tag{4.9}$$

Moreover, again using (3.5),

$$\int_S G(x,y)\varphi(x)dx \geq \frac{\delta(y)}{c_1}\int_S \varphi(x)\delta(x)dx, \quad y \in \Omega. \tag{4.10}$$

Let $\alpha > 0$. Due to the absolute continuity of the integral of $\varphi^q\chi_{_E}$, there exists $\sigma > 0$ such that for each measurable set $F \subset E$:

$$|E \setminus F| < \sigma \Rightarrow \Big(\int_{E\setminus F} \varphi(x)^q dx\Big)^{\frac{1}{q}} < \frac{\alpha}{c_1^3}|\varphi\delta|_{1,E} \Rightarrow c_1^2|\varphi|_{q,(E\setminus F)}\delta(y) < \frac{\alpha}{c_1}|\varphi\delta|_{1,E}\delta(y), \quad y \in \Omega.$$

Using (4.9) and (4.10),

$$\int\limits_{E\setminus F} G(x,y)\varphi(x)dx < \alpha \int\limits_{E} G(x,y)\varphi(x)dx, \quad y \in \Omega.$$

Multiplying by $K(y,z)$ and integrating on Ω with respect to y we get the claim.■

Proof of Theorem 2 Let $F \subset E$ and $\varphi \in L^1_+(E)$. Since $\varphi\chi_{_F} \in L^1_+(E)$, if $\varphi\chi_{_F} \neq 0$, from the definition of $\lambda(E)$, we get

$$\lambda(E) \leq \operatorname*{esssup}_{z\in E\setminus\partial\Omega} \frac{(\varphi\chi_{_F})(z)}{\int\limits_E H(x,z)(\varphi\chi_{_F})(x)dx} = \operatorname*{esssup}_{z\in F\setminus\partial\Omega} \frac{(\varphi\chi_{_F})(z)}{\int\limits_F H(x,z)(\varphi\chi_{_F})(x)dx} = \lambda(F,\varphi\chi_{_F}),$$

then

$$\lambda(E) \leq \inf\left\{\lambda(E,\varphi\chi_{_F}) \,\middle|\, \varphi \in L^1_+(E),\, \varphi\chi_{_F} \neq 0\right\} =$$
$$= \inf\left\{\lambda(F,\varphi) \,\middle|\, \varphi \in L^1_+(F),\, \varphi \neq 0\right\} = \lambda(F).$$

We continue by proving the other estimate stated in the claim.

Let $\alpha > 0$ (since $\lambda(E) < +\infty$, see (4.3)), denote

$$\beta = \frac{\alpha}{1+\lambda(E)+\alpha}.$$

Let $\Phi \in L^q_+(\Omega)$ be such that (see Lemma 4.2)

$$\lambda(E) = \operatorname*{esssup}_{z\in E\setminus\partial\Omega} \frac{\Phi(z)}{\int\limits_E H(x,z)\Phi(x)dx}.$$

By the previous lemma, there exists $\sigma > 0$ such that for each measurable $F \subset E$:

$$|E\setminus F| < \sigma \Rightarrow \int\limits_{E\setminus F} H(x,z)\Phi(x)dx < \beta \int\limits_E H(x,z)\Phi(x)dx, \quad z \in \Omega.$$

Therefore

$$|E\setminus F| < \sigma \Rightarrow \lambda(F) \leq \lambda(F,\varphi\chi_{_F}) = \operatorname*{esssup}_{z\in F\setminus\partial\Omega} \frac{(\Phi\chi_{_F})(z)}{\int\limits_F H(x,z)(\Phi\chi_{_F})(x)dx} =$$

$$= \operatorname*{esssup}_{z\in F\setminus\partial\Omega} \frac{(\Phi\chi_{_F})(z)}{\int\limits_E H(x,z)\Phi(x)dx} \cdot \frac{\int\limits_E H(x,z)\Phi(x)dx}{\int\limits_F H(x,z)\Phi(x)dx} \leq$$

$$\leq \operatorname*{esssup}_{z\in F\setminus\partial\Omega} \lambda(E)\frac{\int\limits_E H(x,z)\Phi(x)dx}{\int\limits_E H(x,z)\Phi(x)dx - \int\limits_{E\setminus F} H(x,z)\Phi(x)dx} =$$

$$= \lambda(E)\operatorname*{esssup}_{z\in F\setminus\partial\Omega} \frac{1}{1-\frac{\int\limits_{E\setminus F} H(x,z)\Phi(x)dx}{\int\limits_E H(x,z)\Phi(x)dx}} \leq \frac{\lambda(E)}{1-\beta}.$$

Due to the definition of β,

$$\lambda(F) \leq \frac{\lambda(E)}{1-\frac{\alpha}{1+\lambda(E)+\alpha}} = \frac{\lambda(E)(1+\lambda(E)+\alpha)}{1+\lambda(E)} \leq \lambda(E)\left(1+\frac{\alpha}{1+\lambda(E)}\right) \leq \lambda(E)+\alpha.$$

Then the proof is done.∎

5. On the integral equation (1.2).

Since $g(z,\cdot)$ is not defined in 0, we search a solution in the limit points of the set of the solutions of the approximate integral equations

$$u(x) = \int\limits_\Omega H(x,z)g(z,\varepsilon+u(z))dz, \quad \varepsilon>0. \tag{5.1}$$

Thanks to $(\mathcal{A}_1)$ and (3.8), there exists a solution $u_\varepsilon \in L^1_+(\Omega)$, $\varepsilon>0$, to (5.1), (see [6, Appendix 2]).

Denoting

$$g_\varepsilon = g(\cdot,\varepsilon+u_\varepsilon),$$

the following statements are consequences of $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and Lemma 3.3.

Lemma 5.1 (boundedness di $(\delta g_\varepsilon)_{\varepsilon>0}$) (see [7, Lemma 5.1]) *Let $E\subset\Omega$ be a measurable set and $0<\varepsilon\leq\frac{1}{4}$. There results*

$$|\delta g_\varepsilon|_{1,E} \leq T(E)^{\frac{p}{p+1}} + T(E),$$

where

$$T(E) = |\delta g^*(\cdot,1/4)|_{1,E} + c_2|\delta^{1-p}\varphi_0|^{\frac{1}{p}}_{1,E},$$

and c_2 is the constant of Lemma 3.3.ii).

Corollary 5.2 (see [7, Lemma 5.2]) *For each $\lambda>0$, there exists $\sigma>0$ such that*

$$|E|<\sigma,\ 0<\varepsilon\leq\frac{1}{4} \Rightarrow |\delta g_\varepsilon|_{1,E}<\lambda.$$

Lemma 5.3 *Let* $\varepsilon > 0$. *There results*

(5.2) $\qquad g_\varepsilon \in L^p(\Omega).$

(5.3) $\qquad \int\limits_\Omega K(\cdot, z) g_\varepsilon(z) dz \in L^q(\Omega).$

Proof. Since $g_\varepsilon \leq g^*(\cdot, \varepsilon)$, (5.2) and (5.3) are consequence of $(\mathcal{A}_1)$ and Lemma 3.2, respectively.■

For the sake of simplicity we fix an increasing sequence

$$(\Omega_n)_{n\in\mathbb{N}^*}, \quad \frac{1}{n} \leq \mathrm{dist}(\Omega_n, \partial\Omega)$$

that covers Ω.

The proof of the following lemma is similar to the one of [7, Lemma 5.4], we simply sketch and improve it.

Lemma 5.4 (convergence) *There exists* $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$, *such that, for each* $n \in \mathbb{N}^*$

$$\Big(\int\limits_{\Omega_n} H(\cdot, z) g_{\varepsilon_k}(z) dz\Big)_{k\in\mathbb{N}}$$

is converging in $L^1(\Omega)$. *Denoting*

$$v_n := \lim_k \int\limits_{\Omega_n} H(\cdot, z) g_{\varepsilon_k}(z) dz,$$

$(v_n)_{n\in\mathbb{N}^*}$ *is increasing and* $v_n \in L^1(\Omega)$, $n \in \mathbb{N}$. *Denoting also*

$$u_0 := \sup_n v_n = \lim_n v_n,$$

there results $u_0 \in L^1_+(\Omega)$ *and*

$$u_{\varepsilon_k} \to u_0 \quad in \;\; L^1(\Omega).$$

Proof. Due to the boundedness of $(\delta g_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega)$, each family $(\chi_{\Omega_n} g_\varepsilon)_{\varepsilon>0}$, $n \in \mathbb{N}$, is bounded in $L^1(\Omega)$. Moreover, due to the compactness of H (see Theorem 3.5), there exists $(\varepsilon_k^1)_{k\in\mathbb{N}^*}$, $\varepsilon_k^1 \to 0$, such that

$$\Big(\int\limits_{\Omega_1} H(\cdot, z) g_{\varepsilon_k^1}(z) dz\Big)_{k\in\mathbb{N}^*}$$

is converging in $L^1(\Omega)$ to some function v_1. There exists $(\varepsilon_k^n)_{k\in\mathbb{N}^*}$, $\varepsilon_k^n \to 0$, subsequence of $(\varepsilon_k^1)_{k\in\mathbb{N}^*}, \cdots, (\varepsilon_k^{n-1})_{k\in\mathbb{N}^*}$, such that

$$\Big(\int\limits_{\Omega_i} H(\cdot,z)g_{\varepsilon_k^n}(z)dz\Big)_{k\in\mathbb{N}^*}, \quad 1\le i\le n,$$

is converging in $L^1(\Omega)$ to some function v_n. Clearly $v_1\le v_2\le\cdots\le v_n$.

Let $(\varepsilon_k)_{k\in\mathbb{N}}$, be the diagonal sequence, it is an extract of each $(\varepsilon_k^n)_{k\in\mathbb{N}}$, it is infinitesimal and

$$v_n=\lim_k\int\limits_{\Omega_n} H(\cdot,z)g_{\varepsilon_k}(z)dz, \quad \text{in } L^1(\Omega), \quad n\in\mathbb{N}^*.$$

$(v_n)_{n\in\mathbb{N}^*}$ is increasing and $v_n\in L^1(\Omega)$. There exists a measurable nonnegative map $u_0:\Omega\to\mathbb{R}$, such that

$$u_0=\operatorname*{esssup}_n v_n=\lim_n v_n, \quad \text{a.e. in } \Omega.$$

Consider

$$u'_{k,n}=\int\limits_{\Omega_n} H(\cdot,z)g_{\varepsilon_k}(z)dz, \quad u''_{k,n}=u_{\varepsilon_k}-u'_{k,n},$$

since

$$\int\limits_{\Omega} u'_{k,n}(x)dx\le\int\limits_{\Omega_n} g_{\varepsilon_k}(z)dz\int\limits_{\Omega} H(x,z)dx,$$

using Lemmas 3.3.*ii*) and Lemma 5.1,

$$\int\limits_{\Omega} u'_{k,n}(x)dx\le c_2\int\limits_{\Omega_n}\delta(z)g_{\varepsilon_k}(z)dz\le c_2\big(T(\Omega)^{\frac{p}{p+1}}+T(\Omega)\big).$$

Due to the definition of v_n, and the Fatou Lemma,

$$\int\limits_{\Omega} v_n(x)dx\le c_2\big(T(\Omega)^{\frac{p}{p+1}}+T(\Omega)\big).$$

By the Beppo Levi Theorem,

$$\int\limits_{\Omega} u_0(x)dx\le c_2\big(T(\Omega)^{\frac{p}{p+1}}+T(\Omega)\big).$$

Hence $u_0\in L^1(\Omega)$. We continue by proving that

$$\lim_k|u_{\varepsilon_k}-u_0|_{1,\Omega}=0.$$

From the Fubini and Tonelli Theorems and Lemma 3.3.ii),

$$\int_{\Omega} u''_{k,n}(x)dx \leq c_2 \int_{\Omega\setminus\Omega_n} \delta(z) g_{\varepsilon_k}(z)dz.$$

Therefore, by Corollary 5.2,

$$\lim_n \int_{\Omega} u''_{k,n}(x)dx = 0,$$

uniformly with respect to k. Let $\sigma > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$n > M_0, \quad k \in \mathbb{N} \Rightarrow \int_{\Omega} u''_{k,n}(x)dx < \sigma. \tag{5.4}$$

Observe that

$$\int_{\Omega} |u_{\varepsilon_k} - u_0|dx \leq \int_{\Omega} |u'_{k,n} - v_n|dx + \int_{\Omega} \big(u_0 - v_n\big)dx + \int_{\Omega} u''_{k,n}dx.$$

Since $\lim_k |u'_{k,n} - v_n|_{1,\Omega} = 0$,

$$n > M_0 \Rightarrow \overline{\lim_k} |u_{\varepsilon_k} - u_0|_{1,\Omega} \leq \int_{\Omega} \big(u_0 - v_n\big)dx + \sigma.$$

Finally, since $u_0 \in L^1(\Omega)$, using the Dominate Convergence Theorem,

$$\overline{\lim_k} |u_{\varepsilon_k} - u_0|_{1,\Omega} \leq \sigma,$$

then $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$. ■

In addition to the upper bound stated in Lemma 5.1, the following statements hold (see [7, (5.6)]).

Lemma 5.5 *There results*

$$\overline{\lim_k} |g_{\varepsilon_k}|_{1,\Omega_n \cap X} \leq c_2 L n^2, \qquad |g(\cdot, u_0)|_{1,\Omega_n \cap X} \leq c_2 L n^2,$$

for each $n \in \mathbb{N}^*$, *where* $X = \{x \in \Omega | u_0(x) \leq L\}$, $L > 0$.

Proof. Let $u'_{k,n}\, u''_{k,n}$ be the ones of the proof of the previous lemma. From Lemma 3.3,

$$u'_{k,n}(x) \geq \frac{\delta(x)}{c_2} \int_{\Omega_n} \delta(z) g_{\varepsilon_k}(z)dz \geq \frac{1}{c_2 n^2} |g_{\varepsilon_k}|_{1,\Omega_n}, \quad x \in \Omega_n.$$

Multiplying by $\dfrac{g_{\varepsilon_k}}{1+u'_{k,n}}$ and integrating on $\Omega_n \cap X$,

$$\text{(5.5)} \qquad \frac{1}{c_2 n^2}|g_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n\cap X} \frac{g_{\varepsilon_k}}{1+u'_{k,n}} dx \leq \int\limits_{\Omega_n\cap X} \frac{u'_{k,n}}{1+u'_{k,n}} g_{\varepsilon_k} dx.$$

Due to the boundedness of $\left(|g_{\varepsilon_k}|_{1,\Omega_n}\right)_{k\in\mathbb{N}^*}$ and Lemma 5.5 in [7],

$$\lim_k |g_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n\cap X} |\frac{1}{1+u'_{k,n}} - \frac{1}{1+v_n}| g_{\varepsilon_k} dx = 0$$

and

$$\lim_k \int\limits_{\Omega_n\cap X} |\frac{u'_{k,n}}{1+u'_{k,n}} - \frac{v_n}{1+v_n}| g_{\varepsilon_k} dx = 0.$$

Hence, from (5.5),

$$\overline{\lim_k} \left(\frac{1}{c_2 n^2}|g_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n\cap X} \frac{g_{\varepsilon_k}}{1+v_n} dx \right) \leq \overline{\lim_k} \int\limits_{\Omega_n\cap X} \frac{v_n}{1+v_n} g_{\varepsilon_k} dx.$$

Reminding that $u_0 = \sup\limits_n v_n$,

$$\frac{1}{1+L}\, \overline{\lim_k}\, |g_{\varepsilon_k}|^2_{1,\Omega_n\cap X} \leq \frac{c_2 n^2 L}{1+L}\, \overline{\lim_k}\, |g_{\varepsilon_k}|_{1,\Omega_n\cap X}.$$

This implies the first estimate of the statement, the second one is consequence of the Fatou Lemma.■

Consequence of these lemmas, as in [7, Theorem 4], is the following fundamental result.

Theorem 5.6 (see [7, Theorem 4]) *Assume* $\mu_0 > \lambda(\Omega_0)$. *There results*

$$u_0 > 0 \text{ a.e. in } \Omega \text{ and } u_0(x) = \int\limits_\Omega H(x,z) g(z, u_0(z)) dz.$$

The last result of this section is the following, that is useful for the next one.

Lemma 5.7 *The following statements hold*

(5.6) $\quad g(\cdot, u_0) \in L^1(\delta, \Omega).$

(5.7) $\quad g_{\varepsilon_k}(\cdot) \to g(\cdot, u_0)$ *in* $L^1(\delta, \Omega).$

(5.8) $$\int_\Omega K(\cdot,z)g(z,u_0(z))dz \in L^1_+(\delta,\Omega).$$

(5.9) $$\int_\Omega K(\cdot,z)g_{\varepsilon_k}(z)dz \to \int_\Omega K(\cdot,z)g(z,u_0(z))dz \quad in \;\; L^1(\delta,\Omega).$$

Proof. (5.6) Since $u_{\varepsilon_k} \to u_0$ a.e. in Ω (see Lemma 5.4) and $u_0 > 0$ a.e. in Ω (see the previous theorem), there results

$$g_{\varepsilon_k} \to g(\cdot,u_0), \quad \text{a.e. in } \Omega.$$

Using Lemma 5.1 and the Fatou Lemma,

$$\int_\Omega \delta(z)g(z,u_0(z))dz \le \underline{\lim}_k \int_\Omega \delta(z)g_{\varepsilon_k}(z)dz \le T(\Omega)^{\frac{p}{p+1}} + T(\Omega),$$

hence (5.6) is done.

Proof. (5.7) If $\operatorname{essinf} u_0 > 0$, due to [6, Lemma 3], (5.7) is trivial. If $\operatorname{essinf} u_0 = 0$, there exists a decreasing family of measurable sets $(X_l)_{l>0}$, $|X_l| > 0$, such that

$$\forall x \in X_l : \quad u_0 \le \frac{1}{1+l}.$$

Observe that

(5.10) $$\int_\Omega \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz \le$$

$$\le \int_{\Omega\setminus X_l} \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz + \int_{\Omega\setminus\Omega_n} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz +$$

$$+ \int_{\Omega_n\cap X_l} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz.$$

Let $\sigma > 0$. By Corollary 5.2 and the absolute continuity of the integral of $\delta g(\cdot,u_0)$, there exists $n \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N} : \quad \int_{\Omega\setminus\Omega_n} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz < \frac{\sigma}{3}.$$

From Lemma 5.5, there exists $l \in \mathbb{N}$ such that

$$\overline{\lim_k} \int_{\Omega_n\cap X_l} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz \le \frac{2c_2n^2}{1+l} < \frac{\sigma}{3}.$$

Hence, from [6, Lemma 3], there exists k_0 such that

$$k > k_0 \ \Rightarrow \int\limits_{\Omega\setminus X_l} \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz < \frac{\sigma}{3}.$$

Therefore, by (5.10),

$$\overline{\lim_{k}} \int\limits_{\Omega} \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz < \sigma.$$

This implies (5.7).

Proof. (5.8) It is consequence of $(\mathcal{A}_2)$ and (5.6).

Proof. (5.9) Since, from $(\mathcal{A}_2)$,

$$\int\limits_{\Omega} \delta(y)dy|\int\limits_{\Omega} K(y,z)\big(g_{\varepsilon_k}(z) - g(z,u_0(z))\big)dz| \le c_0 \int\limits_{\Omega} \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz,$$

the claim follows by (5.7).∎

6. Proof of Theorem 3.

We begin by observing that

$$\nabla_x H(x,z) = \int\limits_{\Omega} \nabla_x G(x,y)K(y,z)dy. \tag{6.1}$$

Let $x_0 \in \Omega$, denote $x = (x_i, x')$, $1 \le i \le n$. There exists $\theta \in]0,1[$ such that

$$\frac{H(x_{0,i}+h, x'_0, z) - H(x_0,z)}{h} = \int\limits_{\Omega} G_{x_i}(x_{0,i}+\theta h, x'_0, y)K(y,z)dy.$$

Since, for each $E \subset \Omega$, by (3.2), we get

$$\int\limits_{E} |G_{x_i}(x_i, x'_0, y)K(y,z)|dy \le$$

$$\le c_1\Big(\int\limits_{E} \frac{1}{\sqrt{(x_i-y_i)^2 + |x'_0 - y'|^2}^{(N-1)q'}}dy\Big)^{\frac{1}{q'}} \cdot \Big(\int\limits_{E} K(y,z)^q dy\Big)^{\frac{1}{q}} \le$$

$$\le c_1|E|^{\frac{1}{rq'}}\Big(\int\limits_{B_D(x_i,x'_0)} \frac{1}{\sqrt{(x_i-y_i)^2 + |x'_0 - y'|^2}^{(N-1)q'r'}}dy\Big)^{\frac{1}{q'r'}} \cdot \Big(\int\limits_{E} K(y,z)^q dy\Big)^{\frac{1}{q}} \le$$

$$\leq c_1|E|^{\frac{1}{rq'}}\Big(\int\limits_{B_D(0)} \frac{1}{|y|^{(N-1)q'r'}}dy\Big)^{\frac{1}{q'r'}} \cdot \Big(\int\limits_E K(y,z)^q dy\Big)^{\frac{1}{q}},$$

where $\frac{N(q-1)}{q-N} < r$ and r' is the conjugate exponent of r. The integral

$$E \mapsto \int\limits_E |G_{x_i}(x_i, x_0', y)K(y,z)|dy$$

is absolutely continuous uniformly with respect to x_i. Using the Vitali Theorem, passing to the limit as $h \to 0$ we get (6.1).

Lemma 6.1 *The following statements hold*

(6.2) $$\int\limits_\Omega \nabla_x H(\cdot, z)g(z, u_0(z))dz \in L^1(\delta, \Omega)^N,$$

(6.3) $$\nabla u_\varepsilon = \int\limits_\Omega \nabla_x H(\cdot, z)g_\varepsilon(z)dz \in L^\infty(\Omega)^N,$$

(6.4) $$\nabla u_{\varepsilon_k} \to \int\limits_\Omega \nabla_x H(\cdot, z)g(z, u_0(z))dz \quad \textit{in } L^1(\delta, \Omega)^N,$$

(6.5) $$\int\limits_\Omega \nabla_x H(\cdot, z)g(z, u_0(z))dz = \nabla u_0 \quad \textit{in the sense of distributions.}$$

Proof. (6.2) By (3.4) and (6.1),

$$I = \int\limits_\Omega \delta(x)dx|\int\limits_\Omega \nabla_x H(x,z)g(z,u_0(z))dz| \leq$$

$$\leq \int\limits_{\Omega\times\Omega\times\Omega} \delta(x)|\nabla_x G(x,y)|K(y,z)g(z,u_0(z))dxdydz \leq$$

$$\leq c_1 \int\limits_{\Omega\times\Omega\times\Omega} \frac{\delta(y)}{|x-y|^{N-1}}K(y,z)g(z,u_0(z))dxdydz.$$

Observe that

(6.6) $$\int\limits_\Omega \frac{dx}{|x-y|^{N-1}} \leq \int\limits_{B_D(y)} \frac{dx}{|x-y|^{N-1}} = \int\limits_{B_D(0)} \frac{dx}{|x|^{N-1}} = \sigma_N D.$$

Hence, from $(\mathcal{A}_2)$,

$$I \le c_1 \int\limits_\Omega g(z,u_0(z))dz \int\limits_\Omega \delta(y)K(y,z)dy \int\limits_\Omega \frac{dx}{|x-y|^{N-1}} \le c_0c_1\sigma_N D \int\limits_\Omega \delta(z)g(z,u_0(z))dz.$$

Therefore, using (5.6), we get (6.2).

Proof. (6.3) Since $g_\varepsilon(z) \le g^*(z,\varepsilon)$, by Lemma 3.2,

$$k_\varepsilon := \int\limits_\Omega K(\cdot,z)g_\varepsilon(z)dz \in L^q(\Omega).$$

Arguing as for (6.1),

$$\nabla u_\varepsilon(x) = \int\limits_\Omega \nabla_x G(x,y)k_\varepsilon(y)dy = \int\limits_\Omega \nabla_x H(x,z)g_\varepsilon(z)dz.$$

Moreover, by (3.2),

$$|\nabla u_\varepsilon(x)| \le \int\limits_\Omega |\nabla_x G(x,y)|k_\varepsilon(y)dy \le c_1 \int\limits_\Omega \frac{k_\varepsilon(y)}{|x-y|^{N-1}}dy \le$$

$$\le c_1|k_\varepsilon|_{q,\Omega}\Big(\int\limits_\Omega \frac{dy}{|x-y|^{(N-1)q'}}\Big)^{\frac{1}{q'}} \le c_1|k_\varepsilon|_{q,\Omega}\left(\int\limits_{B_D(x)} \frac{dy}{|x-y|^{(N-1)q'}}\right)^{\frac{1}{q'}} \le$$

$$\le c_1|k_\varepsilon|_{q,\Omega}\left(\int\limits_{B_D(0)} \frac{dy}{|y|^{(N-1)q'}}\right)^{\frac{1}{q'}}.$$

Since $(N-1)q' < N$, we have that $\nabla u_\varepsilon \in L^\infty(\Omega)^N$.∎

Proof. (6.4) By (6.1), (6.2), (6.3),

$$J = \int\limits_\Omega \delta(x)\big|\nabla u_{\varepsilon_k}(x) - \int\limits_\Omega \nabla_x H(x,z)g(z,u_0(z))dz\big|dx =$$

$$= \int\limits_\Omega \delta(x)\big|\int\limits_\Omega \nabla_x H(x,z)\big(g_{\varepsilon_k}(z) - g(z,u_0(z))\big)dz|dx \le$$

$$\le \int\limits_{\Omega\times\Omega\times\Omega} \delta(x)\big|\nabla_x G(x,z)\big|K(y,z)\big|g_{\varepsilon_k}(z) - g(z,u_0(z))\big|dxdydz.$$

Hence, from (3.4) and $(\mathcal{A}_2)$,

$$J \le c_1 \int\limits_\Omega \big|g_{\varepsilon_k}(z) - g(z,u_0(z))\big| dz \int\limits_\Omega \delta(y) K(y,z) dy \int\limits_\Omega \frac{dx}{|x-y|^{N-1}} \le$$

$$\le c_0 c_1 D \sigma_N \int\limits_\Omega \delta(z) \big|g_{\varepsilon_k}(z) - g(z,u_0(z))\big| dz.$$

Then, (5.7) implies (6.4).

Proof. (6.5) Let $\varphi \in \mathcal{D}(\Omega)$. (6.4) implies

$$\int\limits_\Omega \varphi(x) dx \int\limits_\Omega \nabla_x H(x,z) g(z,u_0(z)) dz =$$

$$= \lim_k \int\limits_\Omega \varphi(x) \nabla u_{\varepsilon_k}(x) dx = -\lim_k \int\limits_\Omega \big(\nabla\varphi(x)\big) u_{\varepsilon_k}(x) dx.$$

Since $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$ (see Lemma 5.4),

$$\int\limits_\Omega \varphi(x) dx \int\limits_\Omega \nabla_x H(x,z) g(z,u_0(z)) dz = -\int\limits_\Omega \nabla\varphi(x) u_0(x) dx,$$

namely (6.5).

Lemma 6.2 *Let $\varepsilon > 0$. There results $u_\varepsilon \in W^{2,q}(\Omega) (\subset C^1(\bar\Omega))$ and*

$$\begin{cases} -\Delta u_\varepsilon(y) = \displaystyle\int_\Omega K(y,z) g\big(z, \varepsilon + u_\varepsilon(z)\big) dz, & \text{for } y \in \Omega, \\ u_\varepsilon(y) = 0, & \text{for } y \in \partial\Omega. \end{cases}$$

Proof. As in the previous lemma denote

$$k_\varepsilon = \int\limits_\Omega K(\cdot, z) g_\varepsilon(z) dz.$$

Since $k_\varepsilon \in L^q(\Omega)$ (see Lemma 5.3), there exists $k_{\varepsilon,n} \in C^\infty(\bar\Omega)$ such that $k_{\varepsilon,n} \to k_\varepsilon$ in $L^q(\Omega)$. Denoting

$$u_{\varepsilon,n}(x) = \int\limits_\Omega G(x,y) k_{\varepsilon,n}(y) dy,$$

due to the regularity of $k_{\varepsilon,n}$,

$$-\Delta u_{\varepsilon,n} = k_{\varepsilon,n}, \quad \text{in } \Omega; \quad u_{\varepsilon,n}\big|_{\partial\Omega} = 0.$$

By [9, Theorem 9.15 and Lemma 9.17], $u_{\varepsilon,n} \in W^{2,q}(\Omega)$ and

$$\|u_{\varepsilon,n} - u_{\varepsilon,m}\|_{W^{2,q}(\Omega)} \leq c|k_{\varepsilon,n} - k_{\varepsilon,m}|_{q,\Omega},$$

with c independent on n and m. Hence,

$$u_\varepsilon \in W^{2,q}(\Omega) \;\text{ e }\; -\Delta u_\varepsilon = k_\varepsilon.$$

Due to the Sobolev Embedding Theorem (see [8, Theorem 5.6]), $u_\varepsilon \in C^1(\bar{\Omega})$. Finally, since

$$u_\varepsilon(x) = \int_\Omega G(x,y)k_\varepsilon(y)dy,$$

we have that $u_\varepsilon\big|_{\partial\Omega} = 0$.■

Proof of Theorem 3 From Lemma 6.1, $u_0 \in W^{1,1}(\delta,\Omega)$. Since $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$, by the Trace Theorem (see [8, pg. 258]), we have that $u_0 \in W_0^{1,1}(\delta,\Omega)$. We prove that u_0 is weak solution to (1.1). Let $\varphi \in \mathcal{D}(\Omega)$. By Lemma 5.4,

$$-\int_\Omega \big(\Delta\varphi(y)\big)u_0(y)dy = -\lim_k \int_\Omega \big(\Delta\varphi(y)\big)u_{\varepsilon_k}(y)dy = -\lim_k \int_\Omega \varphi(y)\Delta u_{\varepsilon_k}(y)dy =$$

$$= \lim_k \int_\Omega \varphi(y)dy \int_\Omega K(y,z)g_{\varepsilon_k}(z)dz.$$

Since $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > 0$, by virtue of (5.9),

$$-\int_\Omega \big(\Delta\varphi(y)\big)u_0(y)dy = \int_\Omega \varphi(y)dy \int_\Omega K(y,z)g(z,u_0(z))dz.$$

The proof is done.■

References

1. D. H. Armitage, S. J. Gardiner, *Classical potential theory.* Springer-Verlag, London, 2001.

2. B. Barceló, L. Escauriaza, E. Fabes, *Boundary regularity of gradients of solutions of elliptic problem. Conference on Partial Differential Equations and Geometry (Torino, 1988).* Rend. Sem. Mat. Univ. Politec. Torino, Special Issue, (1990), 11-24.

3. H. Brezis, X. Cabré, *Some Simple Nonlinear PDE's Without Solutions.* Boll. Un. Mat. Ital. **1-B(8)** (1998), 223-262.

4. M. M. Coclite, *On a Singular Nonlinear Dirichlet Problem - II.* Boll. Un. Mat. Ital. **5-B(7)** (1991), 955-975.

5. M. M. Coclite, *Summable Positive Solutions of a Hammerstein Integral Equation with Singular Nonlinear Term.* Atti Sem. Mat. Fis. Univ. Modena **XLVI** (1998) suppl., 625-639.

6. M. M. Coclite, *Positive Solutions of a Hammerstein Integral Equation with Singular Nonlinear Term.* Topol. Methods Nonlinear Anal. **15** (2000) 2, 235-250.

7. M. M. Coclite, *Positive Solutions of a Hammerstein Integral Equation with Singular Nonlinear Term, II.* to appear on Journal of Integral Equations and Applications.

8. L. C. Evans, *Partial Differential Equations.* American Mathematical Society, Providence, Rhode Island, 1998.

9. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin Heidelberg New York, 1998.

10. S. Gomes, *On a Singular Nonlinear Elliptic Problem,* SIAM J. Math. Anal. **17** (1986), 1359-1369.

11. M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations.* P. Noordhoff, Groningen, The Netherlands, 1964.

12. L. Y. Tsai, *On the Solvability of Integro-differential Equations of Elliptic Type.* Chinese J. Math. **14** (1986) 3, 163-177.

13. L. Y. Tsai, B. Wu, *Existence of Solutions for Elliptic Integro-differential Systems.* Tamkang J. Math. **25** (1994) 1, 61-70.

14. K. Widman, *Inequalities for the Green Function and Boundary Continuity of the Gradient of Solutions of Elliptic Differential Equations.* Math. Scand. **21** (1967), 17-37.

15. X. Zhang, *Boundary Value Problems for Singularly Perturbed Vector Elliptic Integrodifferential Equations..* Acta Math. Sci. **15** (1996) 3, 269-274.

16. Z. Zhao, *Green Function for Schrödinger Operator and Conditioned Feynman-Kac Gauge.* Math. Anal. Appl. **116** (1986), 309-334.